

Infinite-dimensional quantum systems on indefinite causal structures

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Standard quantum mechanics assumes that events are embedded in a global causal structure such that, for every pair of events, the causal order between them is always fixed. The process matrix framework keeps the local validity of standard quantum mechanics while relaxing the assumption on the global causal structure. This allows to describe situations in which the order of events is not fixed, i.e. there are processes where it is not possible to specify whether A causes B or B causes A. Such processes are called causally nonseparable. So far, the formalism has been developed only for finite-dimensional systems and a straightforward generalization to infinite dimensions leads to singularities. Here we develop such generalization, and derive the correlations arising from a causally nonseparable process — the quantum switch — in infinite dimensions. The correlations exhibit interference which is due to superposition of processes in which A is before B and B is before A.

INTRODUCTION

The notion of causality is deeply rooted in our understanding of nature. One can operationally define causality as the possibility of signaling from event A to event B. In ordinary situations with a fixed spacetime background we say that A can cause B if the former belongs to the past light cone of the latter. Equivalently, the effect B belongs to the future light cone of the cause A. Typically, the light cones are fixed, so that it is always possible to apply this notion of causality. Nevertheless, in regimes where both general relativity and quantum mechanics are expected to be relevant it has been argued that the idea of causality should be revised [1–3]. On the one hand, any physical situation described by quantum mechanics can be formulated in terms of a set of probabilities for local outcomes on a global background. On the other hand, in general relativity the causal structure is dynamical, since it depends on the displacements of masses in the spacetime, and probability plays no fundamental role in the theory. Therefore, it has been argued that a theory unifying general relativity and quantum mechanics should be a probabilistic theory on a dynamical causal structure [4]. The process (W -matrix) formalism [5] is an operational way to approach this problem.

The framework retains the validity of ordinary quantum mechanics at a local level, i.e. in local laboratories where quantum operations are performed, whereas no assumptions are made on the global causal structure outside the laboratories. Interestingly, the processes compatible with the W -matrix framework are more general than those allowed by the standard quantum formalism. In particular, they include situations in which the direction of the signaling, and thus causality in the operational sense, is not fixed. Nonetheless, logical paradoxes, such as signaling to the past, are ruled out by consistency conditions. We are interested in processes, which we refer to as *causally nonseparable*, where it is not possible to decompose the W -matrix as [6, 7]

$$W = \lambda W^{A \prec B} + (1 - \lambda) W^{B \prec A}, \quad (1)$$

where $0 \leq \lambda \leq 1$. If equation (1) holds, the W -matrix can always be understood as a classical (convex) mixture of a term which allows signaling from A to B with probability λ and a term which allows signaling from B to A with probability $1 - \lambda$. The possibility for A and B to share an entangled state with no-signaling correlations is also included in equation (1).

One particular causally nonseparable process is the ‘quantum switch’. This is a quantum system with an auxiliary degree of freedom which can coherently control the order in which operations are applied. The quantum switch provides quantum computational advantages with respect to quantum circuits with fixed gate order [8–10] and has recently been implemented with linear optics [11].

The processes studied up to now are limited to finite-dimensional Hilbert spaces [5–7, 10]. In this paper we generalize the formalism to infinite-dimensional Hilbert spaces, where the external degrees of freedom of the system, such as position and momentum, can be investigated. These degrees of freedom have a more direct correspondence with the properties of space and time and provide a link between our operational definition of causality and causality understood in terms of spacetime intervals between events. Moreover, continuous variables are a natural way to pave the path for a formulation of quantum fields on indefinite causal structures.

Here we characterize the W -matrix for infinite-dimensional quantum systems using a phase-space representation in terms of Wigner functions. This approach is suited to deal with infinite dimensions, in contrast to the straightforward

generalization of the approach used in finite dimensions, which leads to singularities as the dimensions go to infinity. We also show that the notion of causal nonseparability is maintained in infinite dimensions and we describe a causally nonseparable process, the quantum switch. Specifically, we provide an argument for the causal nonseparability of the switch in infinite dimensions and show that it exhibits interference due to the superposition of the order in which the operations are applied.

GENERAL FRAMEWORK

We consider two local observers, A and B , each provided with a local laboratory and free to perform local operations on a quantum system. These are represented as completely positive (CP) maps $\mathcal{M}_i^A : \mathcal{L}(\mathcal{H}_{A_1}) \rightarrow \mathcal{L}(\mathcal{H}_{A_2})$, $\mathcal{M}_j^B : \mathcal{L}(\mathcal{H}_{B_1}) \rightarrow \mathcal{L}(\mathcal{H}_{B_2})$, where \mathcal{H}_{X_1} , \mathcal{H}_{X_2} , $X = A, B$, are the (infinite-dimensional) input and output Hilbert spaces of each laboratory and $\mathcal{L}(\mathcal{H})$ denotes the space of linear operators on the Hilbert space \mathcal{H} . Each map \mathcal{M}_i^X describes transformations of a state ρ with outcome i and output state $\mathcal{M}_i^X(\rho)$. A convenient way of representing CP maps is through the Choi-Jamiolkowski (CJ) isomorphism (see [12, 13] for the original definition in finite dimensions, [14] for the extension to infinite dimensions), which associates an operator M_i^X to a CP map \mathcal{M}_i^X through $M_i^X = (\mathbb{1} \otimes \mathcal{M}_i^X) |\Phi^+\rangle \langle \Phi^+|$. Here $|\Phi^+\rangle = \int dx |xx\rangle_{X_1}$ is the non-normalized maximally entangled state in $\mathcal{H}_{X_1} \otimes \mathcal{H}_{X_1}$ and $\mathbb{1}$ is the identity operator. Since the probability of obtaining an outcome is unity, the sum over all possible \mathcal{M}_i^X is a completely positive trace-preserving (CPTP) map. This condition, which we refer to as CPTP condition, is expressed in terms of the CJ equivalent $M^X = \sum_i M_i^X$ as $\text{Tr}_{X_2}(M^X) = \mathbb{1}_{X_1}$.

The process matrix is an operator $W \in \mathcal{L}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2})$ such that $W \geq 0$ and the probability of two measurement outcomes i and j is

$$p(\mathcal{M}_i^A, \mathcal{M}_j^B) = \text{Tr} [W(M_i^A \otimes M_j^B)] . \quad (2)$$

The probability should satisfy $0 \leq p(\mathcal{M}_i^A, \mathcal{M}_j^B) \leq 1$. In particular, the condition $\sum_{ij} p(\mathcal{M}_i^A, \mathcal{M}_j^B) = 1$ implies that $\text{Tr} [W(M^A \otimes M^B)] = 1$ for every pair of CPTP maps M^A, M^B . From now on we will only consider the CJ representation of the CP maps.

In the original formulation [5] in finite-dimensional Hilbert spaces the characterization of the process matrix heavily relies on the dimension of the Hilbert spaces of the observers, so that taking the representation of W and letting the dimensions tend to infinity would lead to singularities. Therefore a straightforward generalization to infinite dimensions is not possible. An alternative formulation, suitable for the case considered, is given in terms of Wigner functions, which provide an equivalent description to the usual operator representation. We will see that the requirement that W gives rise to consistent probabilities restricts the possible Wigner representations.

CHARACTERIZATION OF THE ONE-PARTY SCENARIO

The one party scenario can be obtained from the two parties when the Hilbert spaces of one observer are one-dimensional. The Wigner equivalent of a CPTP map M (we omit here the index relative to the observer) and of a process matrix W is a function of four variables on the phase space, namely $M(\xi_1, \xi_2)$ and $W(\xi_1, \xi_2)$. Here the subscripts 1 and 2 refer respectively to the input and output Hilbert space and the quantity ξ_i corresponds to the point in the phase space $\xi_i = (x_i, p_i)$. In terms of Wigner functions, the CPTP condition becomes $\frac{1}{2\pi} \int d\xi_2 M(\xi_1, \xi_2) = 1$. By computing the Fourier transform $\tilde{M}(\eta_1, \eta_2) = \frac{1}{(2\pi)^2} \int d\xi_1 d\xi_2 M(\xi_1, \xi_2) e^{-i\xi_1 \cdot \eta_1} e^{-i\xi_2 \cdot \eta_2}$, with $\eta_i = (\kappa_i, \omega_i)$, the previous condition reads

$$\tilde{M}(\eta_1, \mathbf{0}) = 2\pi \delta(\eta_1), \quad (3)$$

where $\delta(\eta_i) = \delta(\kappa_i) \delta(\omega_i)$ and δ is the Dirac delta function.

We use the CPTP condition (3) to characterize the W -matrix. In terms of the Wigner representation the normalization of probability $\text{Tr}(WM^A) = 1$ is

$$\frac{1}{(2\pi)^2} \int d\eta_1 d\eta_2 \tilde{W}(\eta_1, \eta_2) \tilde{M}(\eta_1, \eta_2) = 1. \quad (4)$$

For each $\tilde{M}(\eta_1, \eta_2)$ we identify a small interval $S_2(\tilde{M}) \in \mathbb{R}^2$ around $\eta_2 = \mathbf{0}$ where we can approximate $\tilde{M}(\eta_1, \eta_2)$ with $\tilde{M}(\eta_1, \mathbf{0})$. We assume that the function \tilde{M} has a well-defined limit at $\eta_2 = \mathbf{0}$. For all possible $\tilde{M}(\eta_1, \eta_2)$ we choose the smallest interval $S_2 = \min_{\tilde{M}} S_2(\tilde{M})$. We set $S_2 \equiv [-\frac{\epsilon}{2}, \frac{\epsilon}{2}] \times [-\frac{\delta}{2}, \frac{\delta}{2}]$. We now split our integral in two

parts: in the first one the output variables are integrated over S_2 ; in the second one the integration is performed on $\mathbb{R}^2 \setminus S_2$. By using equation (3) in the integral on S_2 , equation (4) reads

$$1 = \frac{\epsilon\delta}{2\pi} \tilde{W}(\mathbf{0}, \mathbf{0}) + \left\langle \tilde{W} \tilde{M} \right\rangle_{\mathbb{R}^2, \mathbb{R}^2 \setminus S_2}, \quad (5)$$

where $\langle f \rangle_{R_i, R_j} = \frac{1}{(2\pi)^2} \int_{R_i} d\boldsymbol{\eta}_1 \int_{R_j} d\boldsymbol{\eta}_2 f(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$. Note that, in order to satisfy equation (5), $\tilde{W}(\boldsymbol{\eta}_1, \mathbf{0})$ can not diverge faster than $1/\epsilon\delta$. This implies that for all possible $\tilde{M}(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$, restricted to the domain $\mathbb{R}^2 \times (\mathbb{R}^2 \setminus S_2)$, the second term in the sum is always equal to the same constant. This can only happen if the second term in the sum in equation (5) vanishes, so we conclude that $\tilde{W}(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2) = 0$ when $\boldsymbol{\eta}_2 \notin S_2$ and $\tilde{W}(\mathbf{0}, \boldsymbol{\eta}_2) = \frac{2\pi}{\epsilon\delta}$ when $\boldsymbol{\eta}_2 \in S_2$. We now send ϵ and δ to zero. In the limit we find

$$\tilde{W}(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2) = 2\pi w(\boldsymbol{\eta}_1) \delta(\boldsymbol{\eta}_2), \quad (6)$$

where $w(\boldsymbol{\eta}_1)$ is a function to be determined.

We now ask which conditions $w(\boldsymbol{\eta}_1)$ should satisfy in order for the probability to be normalized. If we substitute the result (6) in the condition for the normalization of the probability (4) we see that $1 = \frac{1}{2\pi} \int d\boldsymbol{\eta}_1 w(\boldsymbol{\eta}_1) \tilde{M}(\boldsymbol{\eta}_1, \mathbf{0}) = w(\mathbf{0})$. Moreover, we can write the complete expression for the Wigner function as $W(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = \frac{1}{2\pi} \int d\boldsymbol{\eta}_1 e^{i\boldsymbol{\xi}_1 \cdot \boldsymbol{\eta}_1} w(\boldsymbol{\eta}_1)$. The Wigner equivalent of the W -matrix does not depend on the variables of the second Hilbert space. In the operator representation this result is equivalent to having the identity in the second Hilbert space. This is compatible with the finite-dimensional case shown in [5]. Moreover, computing the partial trace on the first system leads to $\text{Tr}_1 W = \frac{1}{(2\pi)^2} \int d\boldsymbol{\xi}_1 d\boldsymbol{\eta}_1 e^{i\boldsymbol{\xi}_1 \cdot \boldsymbol{\eta}_1} w(\boldsymbol{\xi}_1) = w(\mathbf{0}) = 1$. This means that in \mathcal{H}_1 the W -matrix is a state with unit trace. Therefore, the most general form of the total W for the one-party case is $W = \rho \otimes \mathbb{1}$, consistent with the finite-dimensional case.

CHARACTERIZATION OF THE W -MATRIX IN THE TWO-PARTY SCENARIO

In the bipartite case the Wigner equivalent of the W -matrix is a function of eight variables in the phase space $W(\boldsymbol{\xi}_{A_1}, \boldsymbol{\xi}_{A_2}, \boldsymbol{\xi}_{B_1}, \boldsymbol{\xi}_{B_2})$, where the notation is consistent with the previous case. The probability normalization in terms of the Fourier transform of the Wigner equivalents of the operators is

$$1 = \frac{1}{(2\pi)^4} \int d\boldsymbol{\eta}_{A_1} d\boldsymbol{\eta}_{A_2} d\boldsymbol{\eta}_{B_1} d\boldsymbol{\eta}_{B_2} \tilde{W}(\boldsymbol{\eta}_{A_1}, \boldsymbol{\eta}_{A_2}, \boldsymbol{\eta}_{B_1}, \boldsymbol{\eta}_{B_2}) \tilde{M}^A(\boldsymbol{\eta}_{A_1}, \boldsymbol{\eta}_{A_2}) \tilde{M}^B(\boldsymbol{\eta}_{B_1}, \boldsymbol{\eta}_{B_2}), \quad (7)$$

where the CPTP condition for \tilde{M}^A and \tilde{M}^B is described by equation (3). Consider now a specific local operation for one of the two parties, say Alice, given by $\tilde{M}^A(\boldsymbol{\eta}_{A_1}, \boldsymbol{\eta}_{A_2}) = 2\pi \delta(\boldsymbol{\eta}_{A_1}) \chi(R_{A_2})$, where $\chi(R_{A_2})$ is the characteristic function over the set R_{A_2} , $\chi(R_{A_2}) = 1$ when $\boldsymbol{\eta}_{A_2} \in R_{A_2}$, $\chi(R_{A_2}) = 0$ otherwise. R_{A_2} is a two-dimensional set defined as $R_{A_2} = \left[-\frac{1}{2\alpha_1}, \frac{1}{2\alpha_1}\right] \times \left[-\frac{1}{2\alpha_2}, \frac{1}{2\alpha_2}\right]$ and α_1, α_2 are two arbitrary positive numbers. This choice of the measurement satisfies the CPTP condition for all α_1, α_2 . By inserting this in equation (7) we obtain

$$1 = \frac{1}{(2\pi)^3} \frac{\alpha_1 \alpha_2}{\alpha_1 \alpha_2} \int d\boldsymbol{\eta}_{A_2} d\boldsymbol{\eta}_{B_1} d\boldsymbol{\eta}_{B_2} \tilde{W}(\mathbf{0}, \boldsymbol{\eta}_{A_2}, \boldsymbol{\eta}_{B_1}, \boldsymbol{\eta}_{B_2}) \chi(R_{A_2}) \tilde{M}^B(\boldsymbol{\eta}_{B_1}, \boldsymbol{\eta}_{B_2})$$

If we now let α_1, α_2 be very large, but still finite, we can approximate $\alpha_1 \alpha_2 \chi(R_{A_2})$ with the product of two delta functions, so that we can perform the integration in $\boldsymbol{\eta}_{A_2}$ by evaluating the W -matrix in the origin. Therefore, the condition to impose on the total W to have an integral converging to a constant (one) is $\tilde{W}(\mathbf{0}, \boldsymbol{\eta}_{A_2}, \boldsymbol{\eta}_{B_1}, \boldsymbol{\eta}_{B_2}) = 2\pi \alpha_1 \alpha_2 \tilde{W}_B(\boldsymbol{\eta}_{B_1}, \boldsymbol{\eta}_{B_2})$ whenever $\boldsymbol{\eta}_{A_2} \in R_{A_2}$ and $W = 0$ otherwise. $\tilde{W}_B(\boldsymbol{\eta}_{B_1}, \boldsymbol{\eta}_{B_2})$ is the reduced W of the observer B. As a consequence, in the limit $\alpha_1, \alpha_2 \rightarrow \infty$ we obtain $1 = \frac{1}{(2\pi)^2} \int d\boldsymbol{\eta}_{B_1} d\boldsymbol{\eta}_{B_2} \tilde{W}_B(\boldsymbol{\eta}_{B_1}, \boldsymbol{\eta}_{B_2}) \tilde{M}^B(\boldsymbol{\eta}_{B_1}, \boldsymbol{\eta}_{B_2})$. The previous equation describes exactly the one-party case, so we can apply the result (6) and write

$$\tilde{W}(\mathbf{0}, \boldsymbol{\eta}_{A_2}, \boldsymbol{\eta}_{B_1}, \boldsymbol{\eta}_{B_2}) = (2\pi)^2 \tilde{w}_{B_1}(\boldsymbol{\eta}_{B_1}) \delta(\boldsymbol{\eta}_{B_2}) \delta(\boldsymbol{\eta}_{A_2}). \quad (8)$$

This decomposition of \tilde{W} is correct only if $\boldsymbol{\eta}_{A_2}$ is arbitrarily close to the origin. If we now repeat the same procedure by swapping the measurements of Alice and Bob we find an analogous condition

$$\tilde{W}(\boldsymbol{\eta}_{A_1}, \boldsymbol{\eta}_{A_2}, \mathbf{0}, \boldsymbol{\eta}_{B_2}) = (2\pi)^2 \tilde{w}_{A_1}(\boldsymbol{\eta}_{A_1}) \delta(\boldsymbol{\eta}_{A_2}) \delta(\boldsymbol{\eta}_{B_2}), \quad (9)$$

which holds when $\boldsymbol{\eta}_{B_2}$ is arbitrarily close to the origin.

We now go back to the equation (7) for the normalization of probability. Similarly to the one-party case, we define two intervals $S_{A_2} = [-\frac{\epsilon_A}{2}, \frac{\epsilon_A}{2}] \times [-\frac{\delta_A}{2}, \frac{\delta_A}{2}] \in \mathbb{R}^2$ and $S_{B_2} = [-\frac{\epsilon_B}{2}, \frac{\epsilon_B}{2}] \times [-\frac{\delta_B}{2}, \frac{\delta_B}{2}] \in \mathbb{R}^2$, where we can approximate the functions \tilde{M}^A and \tilde{M}^B with their values in respectively $\boldsymbol{\eta}_{A_2} = \mathbf{0}$ and $\boldsymbol{\eta}_{B_2} = \mathbf{0}$. We can now split the probability condition in four parts, writing the integrals over A_2 and B_2 as the sum of an integral over S_{A_2} and S_{B_2} and on the rest of the integration region \bar{S}_{A_2} and \bar{S}_{B_2} . Using the CPTP condition for the local operations we find

$$1 = P_{S_{A_2}, S_{B_2}} + P_{S_{A_2}, \bar{S}_{B_2}} + P_{\bar{S}_{A_2}, S_{B_2}} + P_{\bar{S}_{A_2}, \bar{S}_{B_2}} \quad (10)$$

where

$$\begin{aligned} P_{S_{A_2}, S_{B_2}} &= \text{const}, \\ P_{S_{A_2}, \bar{S}_{B_2}} &= k_A \int d\boldsymbol{\eta}_{B_1} \tilde{w}_{B_1}(\boldsymbol{\eta}_{B_1}) \int_{\mathbb{R}^2 \setminus S_{B_2}} d\boldsymbol{\eta}_{B_2} \delta(\boldsymbol{\eta}_{B_2}), \\ P_{\bar{S}_{A_2}, S_{B_2}} &= k_B \int d\boldsymbol{\eta}_{A_1} \tilde{w}_{A_1}(\boldsymbol{\eta}_{A_1}) \int_{\mathbb{R}^2 \setminus S_{A_2}} d\boldsymbol{\eta}_{A_2} \delta(\boldsymbol{\eta}_{A_2}), \\ P_{\bar{S}_{A_2}, \bar{S}_{B_2}} &= \left\langle \tilde{W} \tilde{M}^A \tilde{M}^B \right\rangle_{\mathbb{R}^2, \mathbb{R}^2, \mathbb{R}^2 \setminus S_{A_2}, \mathbb{R}^2 \setminus S_{B_2}}. \end{aligned}$$

Here, k_A , k_B are two constants and the notation for the last term is analogous to the one used in the one-party case. $P_{S_{A_2}, \bar{S}_{B_2}}$ and $P_{\bar{S}_{A_2}, S_{B_2}}$ are identically zero because the delta functions vanish in the interval. Since the integral is equal to the same constant for all local operations we conclude that the fourth term is zero in the interval considered. To do this, the W -function should be zero outside S_{A_2} or S_{B_2} , at least in one of the outputs. Setting \tilde{W} equal to zero in the input would instead lead to the trivial solution $W = 0$. By taking the limit when the intervals S_{A_2}, S_{B_2} reduce to a point, and following an analogous procedure to the one-party case, it is possible to show that the W -matrix is a delta function at least in one of the two outputs. Applying the inverse Fourier transform, in the original variables $\boldsymbol{\xi}_i$ the conditions on the W imply that the Wigner equivalent of the process matrix can not depend on both outputs at the same time, i.e. $W(\boldsymbol{\xi}_{A_1}, \boldsymbol{\xi}_{A_2}, \boldsymbol{\xi}_{B_1})$ or $W(\boldsymbol{\xi}_{A_1}, \boldsymbol{\xi}_{B_1}, \boldsymbol{\xi}_{B_2})$. As we have already pointed out in the one-party scenario, this condition is equivalent to having an identity in at least one of the two output Hilbert spaces when W is represented in the space of linear operators on the tensor product of the four Hilbert spaces.

To summarize, we have shown that the bipartite W allows for three different situations. The first case is described by a shared state between A and B in the input Hilbert space, but identity in both output Hilbert spaces, i.e. $W = \rho_{A_1 B_1} \otimes \mathbb{1}_{A_2 B_2}$. This corresponds to no-signaling between the two observers. The second and third case describe signaling from one observer to the other. In this case the W -matrix is written as $W_{A_1 B_1 B_2} \otimes \mathbb{1}_{A_2}$ when B signals to A , or as $W_{A_1 A_2 B_1} \otimes \mathbb{1}_{B_2}$ when A signals to B . For further discussion on the specific form of these terms see [5].

QUANTUM SWITCH IN INFINITE DIMENSIONS

A scheme of the quantum switch is provided in Figure 1. The switch involves three local observers, which we denote as A , B and C . The observers perform local quantum operations, here chosen to be a measurement followed by a reparation of a quantum state. Outside the laboratories the system propagates along two “fibers” (solid and dotted line in Figure 1). A quantum state $|\psi_I\rangle$ is prepared at time t_I and sent in a superposition of two paths. In one of the paths the particle enters laboratory A at time t_1 and laboratory B at time $t_2 > t_1$; in the second path the order of the operations A and B is reversed. After exiting the laboratories A and B the system is detected by the observer C at time t_O . Note that in order to preserve the coherence of the process the measurements should not reveal the time.

The switch describes a quantum process in which the order of the local operations is in a superposition. In finite dimensions it has been proved that the W -matrix which describes the switch is causally nonseparable [6], i.e. it can not be written as $W = \lambda W^{A \prec B \prec C} + (1 - \lambda) W^{B \prec A \prec C}$, where C always comes after A and B and $0 \leq \lambda \leq 1$. Here we generalize the switch to infinite dimensions, and provide an alternative proof of its causal nonseparability.

The W -matrix is an operator acting on the tensor product of six Hilbert spaces, $W \in \mathcal{L}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2} \otimes \mathcal{H}_{C_1} \otimes \mathcal{H}_p)$. The first five spaces are infinite-dimensional and \mathcal{H}_p is a two-dimensional Hilbert space spanned by the vectors $|0\rangle$ and $|1\rangle$, which label each of the paths (fibers) taken by the particle (see Figure 1). The W -matrix of the switch is pure and can be written as $W = |w\rangle\langle w|$, where $|w\rangle = \int d\bar{r} w(\bar{r}) |\bar{r}\rangle$, with $\bar{r} = (r_{A_1}, r_{A_2}, r_{B_1}, r_{B_2}, r_{C_1})$. Explicitly,

$$w(\bar{r}) = \frac{1}{\sqrt{2}} \int dr_I \psi_I(r_I) (w^{A \prec B \prec C} |0\rangle + w^{B \prec A \prec C} |1\rangle). \quad (11)$$

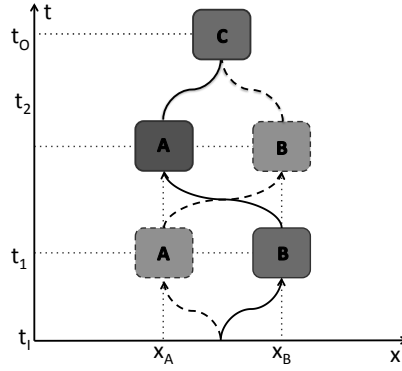


FIG. 1. A quantum system is prepared in a state $|\psi_I\rangle$ at time t_I and is sent in a superposition of two paths. Each path, realized by sending the particle through a fiber (solid and dotted line in the figure), enters the two laboratories A and B in a fixed order and is detected by C at time t_O after exiting the two laboratories. In each local laboratory the state undergoes local quantum operations described as measurement and reparation. The probability of measurement outcomes shows an interference pattern due to the superposition of two causal orders. The interference can not be reproduced from local operations performed in a fixed causal order.

Here, $\psi_I(r_I)$ is a normalized square-integrable function. The variables of the functions $w^{A \prec B \prec C} = w^{A \prec B \prec C}(r_I, \bar{r})$ and $w^{B \prec A \prec C} = w^{B \prec A \prec C}(r_I, \bar{r})$, where the arguments parametrize the propagation along the fiber, are omitted in (11) for simplicity. The total state $|w\rangle$ is a superposition of two terms, described by $w^{A \prec B \prec C}$ and $w^{B \prec A \prec C}$, which can be explicitly written as

$$w^{A \prec B \prec C} = G_{I1}(r_{A1} - r_I) G_{12}(r_{B1} - r_{A2}) G_{2O}(r_{C1} - r_{B2}) \quad (12)$$

$$w^{B \prec A \prec C} = G_{I1}(r_{B1} - r_I) G_{12}(r_{A1} - r_{B2}) G_{2O}(r_{C1} - r_{A2}) \quad (13)$$

where $G_{ab}(r_b - r_a) = \langle r_b | e^{-\frac{i}{\hbar} \hat{H}(t_b - t_a)} | r_a \rangle$ is the Green function between r_a and r_b and \hat{H} is the hamiltonian which generates the evolution along the fiber.

Consider now the local operations performed in A and B. Suppose that each observer measures the state in a region R_i (R_j) of the whole laboratory A (B). Afterwards, the state is reprepared in $|\phi_A\rangle$ ($|\phi_B\rangle$). The Choi-Jamiołkowski equivalent of this local operation in A's laboratory is $M_i^A = \int_{R_i} dy_A |y_A\rangle \langle y_A| \otimes |\phi_A\rangle \langle \phi_A|$. The intervals R_i satisfy $R_i \cap R_j = \emptyset$ for $i \neq j$ and $\cup_i R_i = V_A$, where V_A is the volume of the local laboratory. A similar expression is valid for the case of B. The observer C detects the state he receives by projecting it over the region R_k of the volume of his laboratory V_C and by recombining the two paths via a measurement on the $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ basis. As a consequence, the local operation performed by C is $M_{k\pm}^C = M_k^C \otimes |\pm\rangle \langle \pm|$, where $M_k^C = \int_{R_k} dy_C |y_C\rangle \langle y_C|$ and it is implied that the output Hilbert spaces of C are one-dimensional.

The probability of the measurement outcomes is then given by $p_{ijk\pm} = p(\mathcal{M}_i^A, \mathcal{M}_j^B, \mathcal{M}_{k\pm}^C) = \langle w | (M_i^A \otimes M_j^B \otimes M_{k\pm}^C) | w \rangle$. For simplicity we first consider a density of probability $\Pi_{ijk\pm} = \Pi_{ijk\pm}(r_I, r'_I)$ such that $p_{ijk\pm} = \int dr_I dr'_I \psi_I(r_I) \psi_I^*(r'_I) \Pi_{ijk\pm}(r_I, r'_I)$. Then we can write

$$\Pi_{ijk\pm} = \frac{1}{2} \left[\pi_{ijk\pm}^{A \prec B \prec C} + \pi_{ijk\pm}^{B \prec A \prec C} + 2 \text{Re} \pi_{ijk\pm}^{int} \right], \quad (14)$$

where we can express the single terms in the sum by adopting a vector notation with $|w^{A \prec B \prec C}\rangle = \int d\bar{r} w^{A \prec B \prec C} |\bar{r}\rangle$ and $|w^{B \prec A \prec C}\rangle = \int d\bar{r} w^{B \prec A \prec C} |\bar{r}\rangle$,

$$\begin{aligned} \pi_{ijk\pm}^{A \prec B \prec C} &= \frac{1}{2} \langle w^{A \prec B \prec C} | M_i^A \otimes M_j^B \otimes M_k^C | w^{A \prec B \prec C} \rangle \\ \pi_{ijk\pm}^{B \prec A \prec C} &= \frac{1}{2} \langle w^{B \prec A \prec C} | M_i^A \otimes M_j^B \otimes M_k^C | w^{B \prec A \prec C} \rangle \\ \pi_{ijk\pm}^{int} &= \pm \frac{1}{2} \langle w^{A \prec B \prec C} | M_i^A \otimes M_j^B \otimes M_k^C | w^{B \prec A \prec C} \rangle. \end{aligned} \quad (15)$$

Assuming $t_1 - t_I = t_2 - t_1 = t_O - t_2 = \Delta t$, we can show that $p_{ijk\pm}$ describes a two-way signaling from A to B to C and from B to A to C. Specifically, we show that the two terms $\pi_{ijk\pm}^{A \prec B \prec C}$ and $\pi_{ijk\pm}^{B \prec A \prec C}$ correspond to a process in

which the order of the events is fixed. Instead, $\pi_{ijk\pm}^{int}$ is an interference term, due to the superposition of causal orders, describing a two-way signaling between the three observers. In order to show this we can sum over the outputs of the observers and show how the marginals depend on the settings ϕ_A of M_i^A and ϕ_B of M_j^B .

We assume that the states ψ_I, ϕ_A and ϕ_B are prepared so that the probability of detection in the three local laboratories is almost one. This means that the integration over the volume of any local laboratory (A, B or C) can be extended to an integral over the whole space, since this would amount to adding a negligible term to the sum. Defining $p^{ABC}(ijk \pm | \phi_A, \phi_B) = \int dr_I dr'_I \psi_I(r_I) \psi_I^*(r'_I) \pi_{ijk\pm}^{A \prec B \prec C}$ the integral of the first term in equation (14), we find that $\sum_{jk\pm} p^{ABC}(ijk \pm | \phi_A, \phi_B) = p^{ABC}(i)$, which means that A does not receive information from B and C. Moreover, since $\sum_{ij} p^{ABC}(ijk \pm | \phi_A, \phi_B) = p^{ABC}(k \pm | \phi_B)$, C receives information from B. Finally, the fact that $\sum_{ik\pm} p^{ABC}(ijk \pm | \phi_A, \phi_B) = p^{ABC}(j | \phi_A)$ means that B receives information from A but not from C. Therefore, we conclude that the probability describes a causally ordered process where A signals to B and B signals to C. The situation is symmetric under the exchange of A and B if we consider the integral of the second term in equation (14), $p^{BAC}(ijk \pm | \phi_A, \phi_B) = \int dr_I dr'_I \psi_I(r_I) \psi_I^*(r'_I) \pi_{ijk\pm}^{B \prec A \prec C}$.

A probabilistic mixture of the two terms corresponds to a process with no fixed causal order, but causally separable in the sense previously discussed. In contrast, when the quantum switch is considered an additional interference term appears. The interference corresponds to $\pi_{ijk\pm}^{int}$ in equation (14) and it can be shown to be

$$\begin{aligned} \pi_{ijk\pm}^{int} = & \pm \frac{1}{2} \int_{R_i} dr_{A_1} \int_{R_j} dr_{B_1} \int_{R_k} dr_{C_1} \int dr_{A_2} dr'_{A_2} dr_{B_2} dr'_{B_2} \times \\ & \times w^{A \prec B \prec C}(r_I, \bar{r}) w^{*B \prec A \prec C}(r'_I, \bar{r}') \phi_A(r'_{A_2}) \phi_B^*(r_{A_2}) \phi_B(r'_{B_2}) \phi_B^*(r_{B_2}), \end{aligned} \quad (16)$$

where $w^{A \prec B \prec C} = w^{A \prec B \prec C}(r_I, r_{A_1}, r_{A_2}, r_{B_1}, r_{B_2}, r_{C_1})$ and $w^{B \prec A \prec C} = (r'_I, r_{A_1}, r'_{A_2}, r_{B_1}, r'_{B_2}, r_{C_1})$ were defined in equations (12) and (13). To show that there is two way signaling, we define $p^{int}(ijk \pm | \phi_A, \phi_B) = \int dr_I dr'_I \psi_I(r_I) \psi_I^*(r'_I) \times \pi_{ijk\pm}^{int}$ and sum over the outputs of the three observers. We find that $\sum_{ij} p^{int}(ijk \pm | \phi_A, \phi_B) = p^{int}(k \pm | \phi_A, \phi_B)$, so both A and B signal to C. Moreover, the two conditions $\sum_j p^{int}(ijk \pm | \phi_A, \phi_B) = p^{int}(ik \pm | \phi_A, \phi_B)$ and $\sum_i p^{int}(ijk \pm | \phi_A, \phi_B) = p^{int}(jk \pm | \phi_A, \phi_B)$ mean respectively that B signals to A and C, and A signals to B and C. Therefore, we conclude that there is two-way signaling. Since the W-matrix is pure and the correlations can exhibit signaling in both directions A to B to C and B to A to C, we conclude that the process is causally nonseparable.

We have shown that the W-matrix formalism can be extended to infinite-dimensional Hilbert spaces. We have also implemented an infinite-dimensional version of the quantum switch, deriving correlations which show that the process is causally nonseparable. The probability of measurement outcomes shows an interference due to the causal nonseparability of the process.

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